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SUMMABILITY TESTS FOR SINGULAR POINTS

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1. **Introduction.** King [5] devised two tests for determining when $z=1$ is a singular point of the function $f(z)$ defined by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

having radius of convergence equal to one. The point $z=1$ and radius of convergence one may be chosen without loss of generality.

In this note a theorem is proved which provides necessary and sufficient conditions that $z=1$ be a singular point of the function defined by (1). The corollary to this theorem yields sufficient conditions amenable to calculations since they can be phrased in terms of a well-known summability transform of the sequence of coefficients $\{a_n\}$. Furthermore the theorem extends the results of King [5] and hence of Titchmarsh [8, p. 216] and Hille [4, p. 7].

2. **Results.** Let the infinite matrix $K[\alpha, \beta]=(c_{n,k})$ be defined by

$$c_{00} = 1, \qquad c_{0k} = 0, \qquad k = 1, 2, \ldots.$$

$$\left[\frac{\alpha+(1-\alpha-\beta)z}{1-\beta z}\right]^n = \sum_{k=0}^{\infty} c_{n,k} z^k, \qquad n = 1, 2, \ldots.$$

$K[\alpha, \beta]$ was introduced by Karamata (See [2]) and is the Euler matrix for $K[1-r, 0]=E(r)$, [1]; the Laurent matrix for $K[1-r, r]=S(r)$, [9], and with a slight change the Taylor matrix for $K[0, r]=T(r)$, [3]. (If $T(r)=(c_{nk})$ then $[(1-r)z/(1-rz)]^{n+1}= \sum_{k=0}^{\infty} c_{nk} z^{k+1}$; $n=0, 1, 2, \ldots$)

The following lemma with slight modification is that of Sledd [7]. It is included for completeness.

LEMMA. *If $K[\alpha, \beta]=(c_{n,k})$ for $|\alpha|<1$, $|\beta|<1$ then there exists $\rho>0$, independent of k, such that for $|t|<\rho$ and $k=0, 1, 2, \ldots$*

$$\sum_{n=0}^{\infty} c_{n,k+1} t^n = \frac{(1-\alpha)(1-\beta)t}{(1-\alpha t)^2}\left(\frac{\beta+(1-\alpha-\beta)t}{1-\alpha t}\right)^k.$$

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Proof. Let $f(z)=[\alpha+(1-\alpha-\beta)z]/[1-\beta z]$. If $0<R<1<1/|\beta|$ then there exists $\rho_1>0$ such that if $|t|\leq\rho_1$ and $|z|\leq R$ then $|tf(z)|\leq M<1$. Fix $|t|\leq\rho_1$ and let

$$\phi_t(z)=\frac{1}{1-tf(z)}=\sum_{n=0}^{\infty}t^n[f(z)]^n.$$

Since this convergence is uniform in $|z|\leq R$, one can apply Weierstrass' theorem on uniformly convergent series of analytic functions [6] to write

$$\begin{aligned}\sum_{n=0}^{\infty}t^n[f(z)]^n &= \sum_{n=0}^{\infty}t^n\left[\sum_{k=0}^{\infty}c_{n,k}z^k\right]\\ &= \sum_{k=0}^{\infty}z^k\left[\sum_{n=0}^{\infty}c_{n,k}t^n\right].\end{aligned} \tag{2}$$

But

$$\frac{1}{1-tf(z)}=\frac{1-\beta z}{1-\alpha t}\,\frac{1}{1-\left(\dfrac{\beta+(1-\alpha-\beta)t}{1-\alpha t}\right)z}. \tag{3}$$

There exists $\rho_2>0$ such that $|t|\leq\rho_2$ and $|z|\leq R$ imply

$$|[\beta+(1-\alpha-\beta)t]z/[1-\alpha t]|<1.$$

Thus (3) may be expanded in a power series,

$$\frac{1}{1-tf(z)}=\sum_{k=0}^{\infty}\frac{(1-\beta z)}{(1-\alpha t)}\left(\frac{\beta+(1-\alpha-\beta)t}{1-\alpha t}\right)^k z^k. \tag{4}$$

Then for $|t|\leq\min(\rho_1,\rho_2)$ one has by equating coefficients in (2) and (4) the result of the lemma.

THEOREM 1. *A necessary and sufficient condition that $z=1$ be a singular point of the function defined by the series* (1) *is that*

$$\limsup\left|\sum_{k=0}^{\infty}c_{n,k+1}a_k\right|^{1/n}=1$$

for some $\alpha<1$, $\beta<1$ and $\alpha+\beta>0$ and $(c_{n,k})$ as defined in §2.

Proof. Consider the function

$$F(t)=\frac{(1-\alpha)(1-\beta)t}{(1-\alpha t)^2}f\left(\frac{\beta+(1-\alpha-\beta)t}{1-\alpha t}\right).$$

$F(t)$ is regular in the region D where

$$D=\left\{t:\left|\frac{\beta+(1-\alpha-\beta)t}{1-\alpha t}\right|<1\right\}$$

Furthermore $z=1$ is a singular point of $f(z)$ if and only if $t=1$ is a singular point of $F(t)$. A simple calculation gives

$$D = \begin{cases} t\colon \left| t+\left(\dfrac{\alpha+\beta}{1-\beta-2\alpha}\right)\right| < \left|\dfrac{1-\alpha}{1-\beta-2\alpha}\right|, & 1-\beta-2\alpha > 0 \\ t\colon \operatorname{Re} t < 1, & 1-\beta-2\alpha = 0 \\ t\colon \left| t+\left(\dfrac{\alpha+\beta}{1-\beta-2\alpha}\right)\right| > \left|\dfrac{1-\alpha}{1-\beta-2\alpha}\right|, & 1-\beta-2\alpha < 0. \end{cases}$$

In each case $t=1$ is on the boundary of D and D contains all points of the closed unit disk except $t=1$. Writing $F(t)$ in series form yields

$$F(t) = \frac{(1-\alpha)(1-\beta)t}{(1-\alpha t)^2}\sum_{k=0}^{\infty} a_k\left(\frac{\beta+(1-\alpha-\beta)t}{1-\alpha t}\right)^k$$

provided $t \in D$. By the lemma there exists $\rho>0$ such that for $|t|\le\rho_1<\rho$ and $k=0, 1, 2, \ldots$

$$\sum_{n=0}^{\infty} c_{n,k+1}t^n = \frac{(1-\alpha)(1-\beta)t}{(1-\alpha t)^2}\left(\frac{\beta+(1-\alpha-\beta)t}{1-\alpha t}\right)^k.$$

Since $(1-\alpha)(1-\beta)t/(1-\alpha t)^2$ vanishes for $t=0$ and $[\beta+(1-\alpha-\beta)t]/[1-\alpha t]$ is equal to β for $t=0$, with $|\beta|<1$, there exists $\rho_2(\alpha, \beta)<\rho_1$ such that $|t|\le\rho_2$ implies

$$\left|\sum_{n=0}^{\infty} c_{n,k+1}t^n\right| \le Mr^k \quad \text{for some } r = r(\alpha, \beta) < 1.$$

Thus

$$\left|\sum_{k=0}^{\infty} a_k \sum_{n=0}^{\infty} c_{n,k+1}t^n\right| \le \sum_{k=0}^{\infty}|a_k|\left|\sum_{n=0}^{\infty} c_{n,k+1}t^n\right| = M\sum_{k=0}^{\infty}|a_k|\, r^k$$

which converges since (1) has radius of convergence one. Weierstrass' theorem now implies

$$F(t) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} c_{n,k+1}a_k\right)t^n \quad \text{for } |t| \le \rho_2. \tag{5}$$

By analytic continuation (5) holds in a disk whose boundary contains the singularity of $F(t)$ nearest the origin and $t=1$ is a singular point of $F(t)$ if and only if the radius of convergence of the series (5) is exactly 1, i.e.,

$$\limsup \left|\sum_{k=0}^{\infty} c_{n,k+1}a_k\right|^{1/n} = 1. \tag{6}$$

COROLLARY. *If the sequence* $\{0, a_0, a_1, \ldots\}$ *is* $K[\alpha, \beta]$ *summable* $\alpha<1$, $\beta<1$, $\alpha+\beta>0$ *to a nonzero constant then* $z=1$ *is a singular point of the function given by* (1).

Notice $K[\alpha, \beta]$ is regular for $\alpha<1$, $\beta<1$ and $\alpha+\beta>0$ (See [2]). If $\{b_n\}$ is the $K[\alpha, \beta]$ transform of $\{0, a_0, a_1, \ldots\}$ then $b_0=0$, $b_n=\sum_{k=0}^{\infty} c_{n,k+1}a_k$, $n=1, 2, \ldots$. Now if $\{0, a_0, a_1, \ldots\}$ is $K[\alpha, \beta]$ summable to a nonzero constant then (6) holds.

If the $T(r)$ transform of $\{a_n\}$ is $\{c_n\}$ and the $K[0, r]$ transform of $\{0, a_0, a_1, \ldots\}$ is $\{\gamma_n\}$ then $\gamma_0=0$, $\gamma_n=c_{n-1}(n\geq 1)$ and thus one has immediately the Corollary 2 of [5]. In [1] it is proved that $E(r)$ is translative to the right when $E(r)$ is regular, so the Corollary of the present paper implies Corollary 1 of [5].

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