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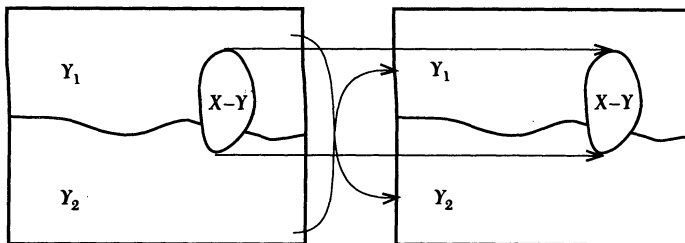


FIGURE 1

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Oscillating Sawtooth Functions

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Ideally every discussion of a theorem in mathematics should contain some comment about its converse. Two important theorems in elementary calculus are:

- (1) If f has a derivative, then f is continuous.
- (2) If f is continuous, then f is a derivative.

Most calculus books contain a number of counterexamples to the converse of (1), but the converse of (2) is usually not discussed in any detail. In this note we will consider a class of functions that can be used to supply counterexamples to the converse of (2).

One approach to developing an example of a discontinuous derivative is to first consider the function $f(x) = x^2 \sin(\frac{1}{x})$ for $x \neq 0$ with $f(0) = 0$. Since $-x^2 \leq f(x) \leq x^2$, the derivative of f at 0 exists and is equal to 0. For $x \neq 0$, the derivative of f is given by $2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$. Thus $f'(x) = g(x) - h(x)$ where $g(x) = 2x \sin(\frac{1}{x})$ for $x \neq 0$ with $g(0) = 0$ and $h(x) = \cos(\frac{1}{x})$ for $x \neq 0$ with $h(0) = 0$. Since $g(x)$ is continuous and $h(x)$ is not continuous, it follows that $f'(x)$ is not continuous. (For another version of this approach see [1] and [2].)

In this note we will consider an alternate approach to finding counterexamples to the converse of (2) that uses "sawtooth functions." Although largely ignored by many calculus texts, "sawtooth functions" are a rich source of interesting examples. In particular, these functions can be used in a direct and elementary way to obtain examples of non-continuous derivatives.

Given a sequence $\{x_k\}$ with $x_k > x_{k+1} > 0$ for all k such that $\lim x_k = 0$, the *oscillating sawtooth function* $f(x)$ associated with $\{x_k\}$ is first defined on each subinterval $[x_{n+1}, x_n]$. The graph of f on $[x_{n+1}, x_n]$ consists of the sides of an isosceles triangle of height one. These triangles are taken to be above the x -axis if n is odd and below the x -axis if n is even. The definition of f is then extended to the entire real line by setting $f(0) = 0$, $f(x) = 0$ if $x > x_1$, and $f(x) = f(-x)$ if $x < 0$.

In more detail, for each k , let $f(x_k) = 0$ and $f(\bar{x}_k) = (-1)^{k+1}$ where $\bar{x}_k = (x_k + x_{k+1})/2$. Extend f to $(0, x_1]$ by connecting $(x_k, 0)$ to $(\bar{x}_k, (-1)^{k+1})$ and $(\bar{x}_k, (-1)^{k+1})$ to $(x_{k+1}, 0)$ by straight lines for each k . For example, if $\bar{x}_k < x < x_k$, then $f(x) = (-1)^{k+1}(1 + (x - \bar{x}_k)/(\bar{x}_k - x_k))$. Next extend f to $[0, \infty)$ by setting $f(0) = 0$ and $f(x) = 0$ for $x > x_1$. Finally extend f to the entire real line by setting $f(x) = f(-x)$ for $x < 0$.

Example 1. To illustrate a specific example of a sawtooth function that gives a counterexample to the converse of (2), we let f denote the oscillating sawtooth function corresponding to $x_n = 1/n$. A partial picture of the graph of f is given in FIGURE 1.

If f were continuous, then by the Fundamental Theorem of Calculus an antiderivative of f would be given by $F(x) = \int_0^x f(t) dt$ for $x \geq 0$ and $F(x) = F(-x)$ for $x < 0$. Since f is not continuous at 0, to show F is an antiderivative of f we must show that $F'(0)$ exists and $F'(0) = f(0) = 0$. By symmetry this reduces to showing that

$$\lim_{x \rightarrow 0^+} \frac{F(x)}{x} = 0.$$

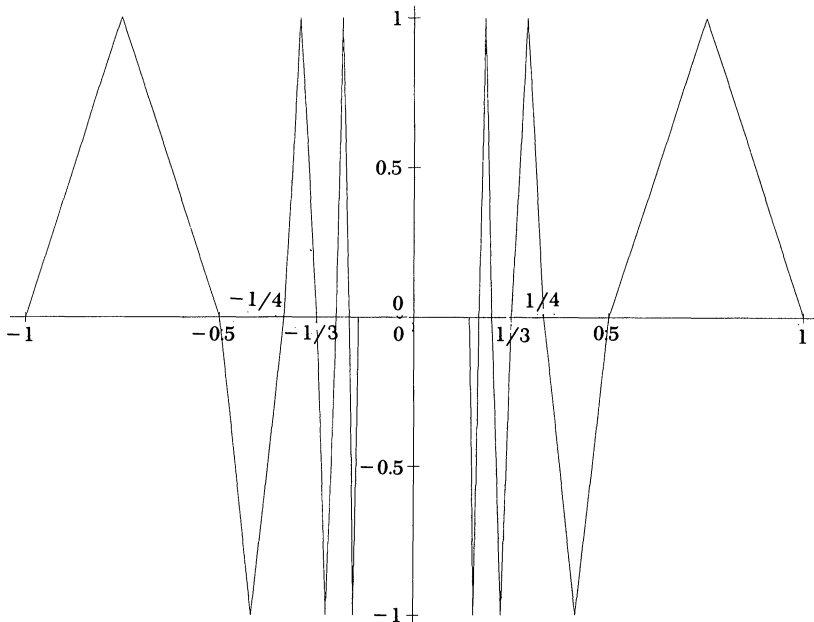


FIGURE 1

If $x_n = 1/n$, then $F(x_n) = F(1/n) = \int_0^{1/n} f(t) dt = \sum_{k=n}^{\infty} (-1)^{k+1} \frac{1}{2} \left(\frac{1}{k} - \frac{1}{k+1} \right)$ where $\frac{1}{2} \left(\frac{1}{k} - \frac{1}{k+1} \right)$ is the area of the isosceles triangle of height one that corresponds to the portion of the graph of f between $x_{k+1} = 1/(k+1)$, and $x_k = 1/k$. Note that the sign is negative when k is even since in this case the corresponding triangle is below the x -axis.

Since $F(1/n)$ is represented by the alternating series

$$\sum_{k=n}^{\infty} (-1)^{k+1} \frac{1}{2} \frac{1}{k(k+1)},$$

we have $|F(1/n)| \leq (1/2)[1/n(n+1)]$. For arbitrary x , there are two possibilities for $F(x)$. If x is in the interval $[1/(n+1), 1/n]$ where n is odd, then $0 \leq F(x) \leq F(1/n)$. If n is even, then $F(1/n) \leq F(x) \leq 0$. In either case, $|F(x)| \leq |F(1/n)|$. Since $x \geq 1/(n+1)$, we have

$$\left| \frac{F(x)}{x} \right| \leq \left| \frac{F(1/n)}{1/(n+1)} \right| \leq \frac{1}{2n},$$

which implies

$$\lim_{x \rightarrow 0^+} \frac{F(x)}{x} = 0.$$

Essentially the same argument as given above can be used to show that the oscillating sawtooth function corresponding to $x_n = 1/n^r$ for any $r > 0$ is a discontinuous derivative. However, not every oscillating sawtooth function can be used as a counterexample to the converse of (2) as can be seen by the following example.

Example 2. Let f be the oscillating sawtooth function corresponding to $x_n = 1/2^n$. This function does not have an antiderivative since in this case we have

$$\begin{aligned} F(x_n)/x_n &= 2^n \sum_{k=n}^{\infty} (-1)^{k+1} \frac{1}{2} \left(\frac{1}{2^k} - \frac{1}{2^{k+1}} \right) \\ &= 2^n \sum_{k=n}^{\infty} (-1)^{k+1} \frac{1}{2^{k+2}} = \frac{(-1)^{n+1} 2^n}{3 \cdot 2^{n+1}} = \frac{(-1)^{n+1}}{6}, \end{aligned}$$

so the limit as $n \rightarrow \infty$ does not exist. Hence $F'(0)$ does not exist.

The above examples are valuable to calculus students not only as a means of gaining insight into the Fundamental Theorem of Calculus and its converse, but also as a way of showing how basic concepts involving the integral, infinite series and the definition of the derivative can be blended together to determine the behavior of certain types of functions.

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