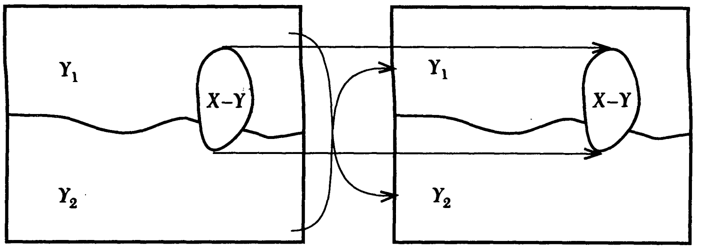

FIGURE 1

REFERENCES

1. P. Halmos, *Naive Set Theory*, Springer-Verlag New York, 1974.
2. I. Kaplansky, *Set Theory and Metric Spaces*, Allyn and Bacon, Boston, 1975.
3. W. Sierpiński, *Cardinal and Ordinal Numbers*, Państwowe Wydawnictwo Naukowe, Warszawa, 1958.
4. L. E. Sigler, *Exercises in Set Theory*, Springer-Verlag New York, 1976.

Oscillating Sawtooth Functions

FREDERICK HARTMANN
DAVID SPROWS
Villanova University
Villanova, PA 19085

Ideally every discussion of a theorem in mathematics should contain some comment about its converse. Two important theorems in elementary calculus are:

(1) If f has a derivative, then f is continuous.
(2) If f is continuous, then f is a derivative.

Most calculus books contain a number of counterexamples to the converse of (1), but the converse of (2) is usually not discussed in any detail. In this note we will consider a class of functions that can be used to supply counterexamples to the converse of (2).

One approach to developing an example of a discontinuous derivative is to first consider the function $f(x)=x^2\sin(\frac{1}{x})$ for $x\neq 0$ with $f(0)=0$. Since $-x^2\le f(x)\le x^2$, the derivative of f at 0 exists and is equal to 0. For $x\neq 0$, the derivative of f is given by $2x\sin(\frac{1}{x})-\cos(\frac{1}{x})$. Thus $f'(x)=g(x)-h(x)$ where $g(x)=2x\sin(\frac{1}{x})$ for $x\neq 0$ with $g(0)=0$ and $h(x)=\cos(\frac{1}{x})$ for $x\neq 0$ with $h(0)=0$. Since $g(x)$ is continuous and $h(x)$ is not continuous, it follows that $f'(x)$ is not continuous. (For another version of this approach see [**1**] and [**2**].)

In this note we will consider an alternate approach to finding counterexamples to the converse of (2) that uses "sawtooth functions." Although largely ignored by many calculus texts, "sawtooth functions" are a rich source of interesting examples. In particular, these functions can be used in a direct and elementary way to obtain examples of non-continuous derivatives.

Given a sequence $\{x_k\}$ with $x_k > x_{k+1} > 0$ for all k such that $\lim x_k = 0$, the *oscillating sawtooth function* $f(x)$ associated with $\{x_k\}$ is first defined on each subinterval $[x_{n+1}, x_n]$. The graph of f on $[x_{n+1}, x_n]$ consists of the sides of an isosceles triangle of height one. These triangles are taken to be above the x-axis if n is odd and below the x-axis if n is even. The definition of f is then extended to the entire real line by setting $f(0) = 0$, $f(x) = 0$ if $x > x_1$, and $f(x) = f(-x)$ if $x < 0$.

In more detail, for each k, let $f(x_k) = 0$ and $f(\bar{x}_k) = (-1)^{k+1}$ where $\bar{x}_k = (x_k + x_{k+1})/2$. Extend f to $(0, x_1]$ by connecting $(x_k, 0)$ to $(\bar{x}_k, (-1)^{k+1})$ and $(\bar{x}_k, (-1)^{k+1})$ to $(x_{k+1}, 0)$ by straight lines for each k. For example, if $\bar{x}_k < x < x_k$, then $f(x) = (-1)^{k+1}(1 + (x - \bar{x}_k)/(\bar{x}_k - x_k))$. Next extend f to $[0, \infty)$ by setting $f(0) = 0$ and $f(x) = 0$ for $x > x_1$. Finally extend f to the entire real line by setting $f(x) = f(-x)$ for $x < 0$.

Example 1. To illustrate a specific example of a sawtooth function that gives a counterexample to the converse of (2), we let f denote the oscillating sawtooth function corresponding to $x_n = 1/n$. A partial picture of the graph of f is given in FIGURE 1.

If f were continuous, then by the Fundamental Theorem of Calculus an antiderivative of f would be given by $F(x) = \int_0^x f(t)\,dt$ for $x \geq 0$ and $F(x) = F(-x)$ for $x < 0$. Since f is not continuous at 0, to show F is an antiderivative of f we must show that $F'(0)$ exists and $F'(0) = f(0) = 0$. By symmetry this reduces to showing that

$$\lim_{x \to 0+} \frac{F(x)}{x} = 0.$$

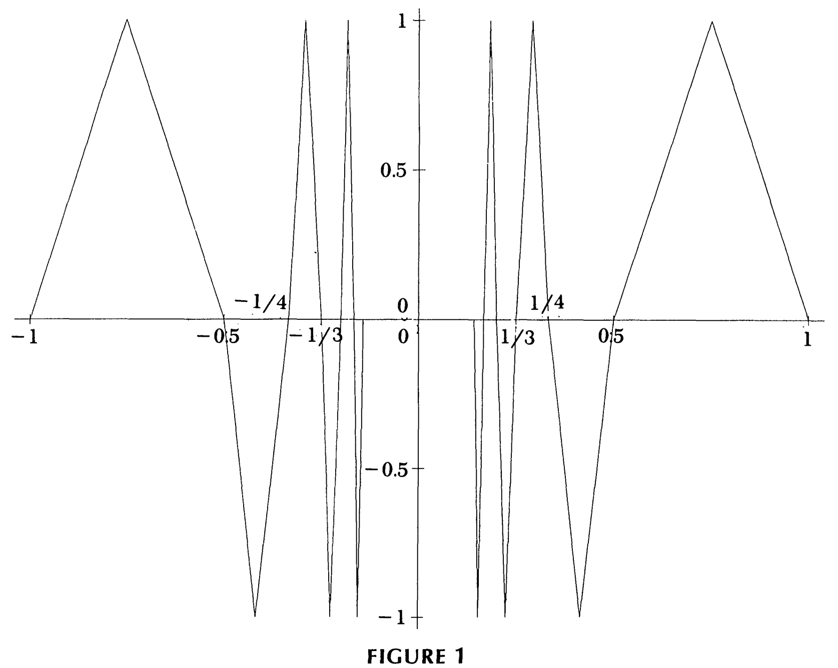

FIGURE 1

If $x_n = 1/n$, then $F(x_n) = F(1/n) = \int_0^{1/n} f(t)\,dt = \sum_{k=n}^{\infty} (-1)^{k+1} \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+1}\right)$ where $\frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+1}\right)$ is the area of the isosceles triangle of height one that corresponds to the portion of the graph of f between $x_{k+1} = 1/(k+1)$, and $x_k = 1/k$. Note that the sign is negative when k is even since in this case the corresponding triangle is below the x-axis.

Since $F(1/n)$ is represented by the alternating series

$$\sum_{k=n}^{\infty}(-1)^{k+1}\frac{1}{2}\frac{1}{k(k+1)},$$

we have $|F(1/n)| \leq (1/2)[1/n(n+1)]$. For arbitrary x, there are two possibilities for $F(x)$. If x is in the interval $[1/(n+1), 1/n]$ where n is odd, then $0 \leq F(x) \leq F(\frac{1}{n})$. If n is even, then $F(\frac{1}{n}) \leq F(x) \leq 0$. In either case, $|F(x)| \leq |F(\frac{1}{n})|$. Since $x \geq 1/(n+1)$, we have

$$\left|\frac{F(x)}{x}\right| \leq \left|\frac{F(\frac{1}{n})}{1/(n+1)}\right| \leq \frac{1}{2n},$$

which implies

$$\lim_{x \to 0+}\frac{F(x)}{x} = 0.$$

Essentially the same argument as given above can be used to show that the oscillating sawtooth function corresponding to $x_n = 1/n^r$ for any $r > 0$ is a discontinuous derivative. However, not every oscillating sawtooth function can be used as a counterexample to the converse of (2) as can be seen by the following example.

Example 2. Let f be the oscillating sawtooth function corresponding to $x_n = 1/2^n$. This function does not have an antiderivative since in this case we have

$$F(x_n)/x_n = 2^n\sum_{k=n}^{\infty}(-1)^{k+1}\frac{1}{2}\left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right)$$

$$= 2^n\sum_{k=n}^{\infty}(-1)^{k+1}\frac{1}{2^{k+2}} = \frac{(-1)^{n+1}2^n}{3 \cdot 2^{n+1}} = \frac{(-1)^{n+1}}{6},$$

so the limit as $n \to \infty$ does not exist. Hence $F'(0)$ does not exist.

The above examples are valuable to calculus students not only as a means of gaining insight into the Fundamental Theorem of Calculus and its converse, but also as a way of showing how basic concepts involving the integral, infinite series and the definition of the derivative can be blended together to determine the behavior of certain types of functions.

REFERENCES

1. J. Klippert, On the right-hand derivative of a certain integral function, *Amer. Math Monthly* 98 (1991), 751–752.
2. S. Ricci, Solution to proposed problem problem 1112, this MAGAZINE 55 (1982), 48.