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The American Mathematical Monthly, Vol. 105, No. 6 (Jun. - Jul., 1998), 508-522.

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The American Mathematical Monthly is currently published by Mathematical Association of America.

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1. INTRODUCTION. Let $P_n(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n$ and

$$T_n(\theta) = \sum_{k=0}^{n} (\lambda_k \cos(k\theta) + \mu_k \sin(k\theta)), \tag{1}$$

where $a_k, \lambda_k, \mu_k \in \mathbb{R}$. How can a_k be chosen so that $P_n(z)$ is univalent on $\{z: |z| < 1\}$? How can λ_k, μ_k be chosen so that $T_n(\theta) \geq 0$ for $\theta \in [0, \pi]$ or $\theta \in [-\pi, \pi]$? Could one easily decide if a given $P_n(z)$ is univalent, or a given $T_n(\theta)$ is non-negative?

Of all these questions, the second is probably easiest to answer. Various positive kernels, for example the Fejér kernel and the Poisson kernel,

$$K_n(\theta) = \frac{1}{2} + \sum_{k=1}^{n} \frac{n-k+1}{n+1} \cos(k\theta), \tag{2}$$

$$P(r, \theta) = 1 + 2\sum_{n=1}^{\infty} r^n \cos(n\theta) = \frac{1-r^2}{1 - 2r\cos\theta + r^2}, \tag{3}$$

both used in Fourier series, come readily to mind. In **[14]** Fejér showed that a trigonometric polynomial of the form (1) is non-negative on $[0, 2\pi]$ if and only if $T_n(\theta) = |h(e^{i\theta})|^2$, for $h(z) = \sum_{k=1}^{n} c_k z^k$, $c_k \in \mathbb{C}$ (if $\mu_k = 0$ for all k, then $c_k \in \mathbb{R}$). Still this characterization does not make it easy to decide if a given trigonometric sum is non-negative, and the univalence of a polynomial isn't assured by any obvious condition.

In this paper we explore many surprising ways in which these two questions are related. It turns out that examples of and facts about non-negative trigonometric sums can be used to produce many interesting examples of univalent polynomials. Conversely, given univalent polynomials can be made to generate non-negative trigonometric sums. Much of this material has not been in the forefront of typical courses in complex and real analysis, and this is a shame, for there is much to gain by studying some of the simple and elegant arguments in this material. We present a variety of examples of this interplay, aided where appropriate by computer algebra computations and graphics for the polynomials. Some of the results cited are valid for general infinite series, but highlighting the results for polynomials allows us to generate interesting graphics.

In what follows $D = \{z: |z| < 1\}$, and the polynomials $P_n(z) = \sum_{k=1}^{n} a_k z^k$ always have $a_k \in \mathbb{R}$ with $a_1 = 1$.

2. FEJÉR AND ALEXANDER. In 1910 Leopold Fejér conjectured that the partial sums of the trigonometric series $\sum_{n=1}^{\infty} \sin(n\theta)/n$, which converges to the positive function $(\pi - \theta)/2$ on $[0, \pi]$, are themselves all positive throughout $[0, \pi]$. This conjecture was proved by Jackson **[24]** and Gronwall **[21]** shortly thereafter, and by many mathematicians in many ways since; see [**2**] for much information on this series and its consequences. In 1915 Alexander [**1**] published a paper on univalent polynomials in which he proved, among other things, that the partial sums of the series $\sum_{n=1}^{\infty} z^n/n$, which represents on D the univalent function $\log(1/(1-z))$, are themselves univalent. There is a connection between these results: if we assume the univalence of $\sum_{n=1}^{N} z^n/n$, then it would seem reasonable, given the symmetry of these partial sums about the real axis, that $\Im\{\sum_{n=1}^{N} z^n/n\} \geq 0$ for $z = re^{i\theta}$, $0 \leq \theta \leq \pi$, $0 \leq r \leq 1$; taking $z = e^{i\theta}$ thus gives $\sum_{n=1}^{N} \sin(n\theta)/n \geq 0$ for $0 \leq \theta \leq \pi$. Thus, a fact about the univalence of a particular polynomial might yield some information about the non-negativity of a trigonometric sum. This intuition can be made precise, as pointed out in [**4**, p. 9], by invoking a result of Dieudonné's [**11**, p. 310]:

$$f(z) = \sum_{n=1}^{N} a_n z^n \text{ is univalent in } D \text{ if and only if} \tag{4}$$

$$2i\frac{f(ze^{i\phi}) - f(ze^{-i\phi})}{ze^{i\phi} - ze^{-i\phi}} = \sum_{n=1}^{N} a_n z^{n-1} \sin(k\phi)/\sin(\phi) \neq 0$$

$$\text{for all } z \in D, 0 \leq \phi \leq \pi.$$

Thus if we assume Alexander's result, and take $a_n = 1/n$ and z real, $z \to 1$, it follows that $\sum_{n=1}^{N} \sin(n\phi)/n > 0$.

This implication is an early instance of the interactions between univalent polynomials and non-negative trigonometric sums that occur in many varied forms throughout the century. Let us look a bit more closely at the Fejér/Alexander interplay: consider the pair $f_N(z) = \sum_{n=0}^{N} z^{2n+1}/(2n+1)$ and $t_N(\theta) = \sum_{n=0}^{N} \sin((2n+1)\theta)/(2n+1)$; f_N is the partial sum of the Taylor series of $\frac{1}{2}\log((1+z)/(1-z))$, which is univalent on D. Alexander considers arg[tangent vector to $f_N(e^{i\phi})$] $= \arg[ie^{i\phi}f_N'(e^{i\phi})] = \pi/2 + \phi + \arg C_N + \sum_k \arg(e^{i\phi} - w_k)$ where $\{w_k\}$ are the critical points of f_N, namely $\{e^{il\pi/(N+1)}\}_{l=1}^{2N+2} \setminus \{1, -1\}$, and C_N a constant. This argument is an increasing function of ϕ, which increases by π between successive critical points in the upper or lower half plane, and by 2π between an upper and lower plane critical point. Thus, for example, the image of $A = \{z: z = e^{i\phi},\ 0 \leq \phi \leq \pi\}$ under f_N begins, when $e^{i\phi} = 1$, as a curve whose right extremity, $f(1)$, is positive and continues moving upward in the pure imaginary direction (by conformality) tracing out a concave-down arc moving to the left. The arc terminates with arg[tangent vector to $f_N(e^{i\phi})$] $= 3\pi/2$ when $e^{i\phi}$ encounters the first critical point of f_N on A. Between this critical point and the next a similar concave-down arc is produced, and the image of the lower portion of the unit circle produces a mirror image of such arcs in the lower half plane. An arc coming from a segment between two critical points in A can't dip below the real axis, for then symmetry would produce a corresponding curve in the upper half plane, which, together with the first arc, yields a closed *clockwise* oriented circuit, violating analyticity. Thus $f_N(A)$ is contained in the upper half plane, $f_N(\overline{A})$ is contained in the lower half plane, and f_N is univalent. Since univalence of an analytic function on the boundary of its domain implies univalence on the domain itself [**28**, p. 425], it follows as before that $t_N(\theta) \geq 0$. See Figure 1.

Fejér managed to reverse this reasoning, passing from the trigonometric sums to univalence. In [**16**], a lecture given at several universities in the U. S. that surveyed some of his results on Fourier series, he noted Alexander's example and offered an alternative proof of univalence: writing $f_N(e^{i\theta}) = u(\theta) + iv(\theta)$ where $u(\theta) =$

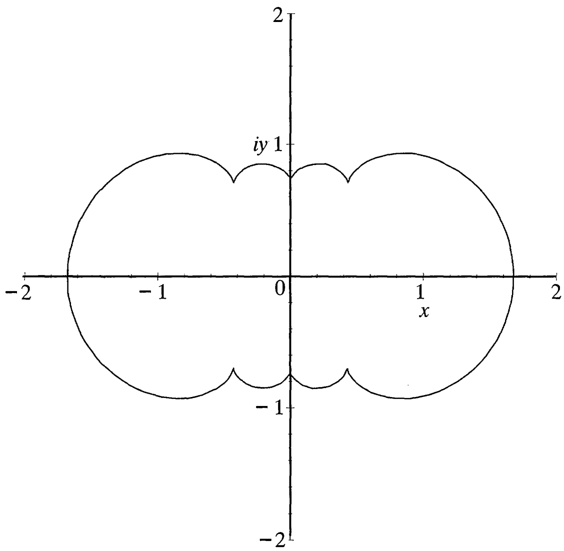


Figure 1. $P(z) = z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \frac{1}{7}z^7$.

$\Sigma_{n=1}^{N}\cos((2n+1)\theta)/(2n+1)$ and $v(\theta) = \Sigma_{n=1}^{N}\sin((2n+1)\theta)/(2n+1)$, he claimed as "well known" that $v(\theta) \geq 0$, $0 \leq \theta \leq \pi$, and pointed out that $\partial u(e^{i\theta})/\partial\theta = -\Sigma_{n=1}^{N}\sin(2n+1)\theta \leq 0$, $0 \leq \theta \leq \pi$, and thus that $f_N(e^{i\theta})$, $0 \leq \theta \leq \pi$, traces out a curve whose real part is monotonically decreasing and lies in the upper half plane. These two facts together with symmetry of the map with respect to the real axis allowed Fejér to conclude that the image of the unit circle under f_N is a Jordan curve, and hence f_N is univalent.

More generally, Fejér considered the class of functions $f = u + iv$ analytic in $\overline{D}$ such that (i) f has real coefficients, (ii) $\partial u(e^{i\theta})/\partial\theta \leq 0$, $0 \leq \theta \leq \pi$, and (iii) $v(e^{i\theta}) \geq 0$, $0 \leq \theta \leq \pi$. Fejér's argument shows that these functions are univalent on D. Functions in this class are said to be "convex in the vertical direction" for the obvious geometric reasons. In a later paper **[18]** Fejér realized that the condition $v(e^{i\theta}) \geq 0$ is redundant, and in **[19]** two rigorous proofs of the univalence of these vertically convex functions are given that use (i) and (ii) alone. Furthermore, since $\partial u/\partial\theta = 0$ on the real axis, it is clear that harmonicity of $\partial u/\partial\theta$ implies that the images of $u(e^{i\theta})$ and $u(re^{i\theta})$, $0 < r < 1$, $0 \leq \theta \leq \pi$, share the same "continual movement to the left" property. In addition, we may speak of a function analytic only on D as "vertically convex" if (i) is satisfied and (ii) $\partial u(re^{i\theta})/\partial\theta \leq 0$, $0 \leq \theta \leq \pi$, for all $r \in (0,1)$; such functions are clearly univalent in D. Fejér and Szegö **[19]** point out that the univalence of a vertically convex function can be formulated as a fact about sine series alone, namely that $\Sigma_k c_k \sin(k\theta) \geq 0$ implies $\Sigma_k c_k \sin(k\theta)/k \geq 0$, $0 \leq \theta \leq \pi$.

It is easy and fun to generate examples of polynomial members of the class. For example, $P_N(z) = \Sigma_{n=1}^{N-1} z^n/n + z^N/2N$ is in the class; this follows since $\partial u(e^{i\theta})/\partial\theta = -\Sigma_{n=1}^{N-1}\sin(n\theta) + (-\sin(N\theta))/2 = -\frac{1}{2}[\Sigma_{n=1}^{N-1}\sin(n\theta) + \Sigma_{n=1}^{N}\sin(n\theta)] = -\sin^2(N\theta/2)\cot(\theta/2) \leq 0$. See Figure 2.

Another easy example, considering Fejér's 1910 conjecture, is $\Sigma_{n=1}^{N} z^n/n^2$, since $\partial u/\partial\theta(e^{i\theta}) = -\Sigma_{n=1}^{N}\sin(\theta)/n \leq 0$ for $\theta \in [0, \pi]$. Let us now look at more ways in which non-negative trigonometric sums can produce facts about the vertically convex class.

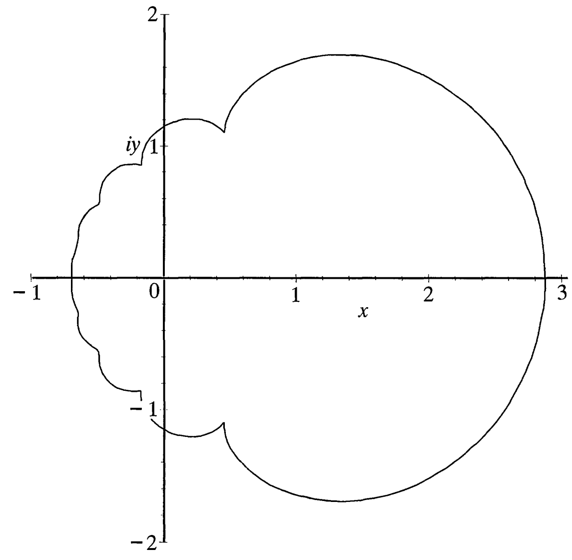


Figure 2. $P_{10}(z)$.

3. FEJÉR'S "THIRD MEAN" AND "FOURFOLD MONOTONE" THEOREMS. In 1927 Szegö [**39**] proved that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is univalent in D then $P_N(z) = \sum_{n=0}^{N} a_n z^n$ is univalent in $\{z: |z| < 1/4\}$, $N = 1, 2, \ldots$. Clearly the partial sums of the Taylor series of a function univalent in D need not also be univalent throughout the disk: $f(z) = z/(1 - z)$ provides an easy counterexample. In 1933 Fejér [**15**] considered beginning with the vertically convex class and seeing if smoothing the partial sums would produce polynomial approximations univalent in D. He succeeded in proving that the third Cesàro mean of a vertically convex function is vertically convex in D, and his proof provides additional ways in which non-negative trigonometric sums contribute to the generation of univalent polynomials. Before we sketch this proof, let us recall some summability facts.

Given an infinite series $\sum_{k=0}^{\infty} u_k$, we define $s_n^{(0)} = \sum_{k=0}^{n} u_k$, $s_n^{(1)} = \sum_{k=0}^{n} s_k^{(0)}$, and in general $s_n^{(N)} = \sum_{k=0}^{n} s_k^{(N-1)}$; s_n^N is the N-th order partial sum of index n. We define $S_n^N = s_n^N / \binom{N+n}{N}$, the *Cesàro mean of order N and index n*. A classical way of generating the $s_n^{(N)}$ is to note that for $0 < r < 1$, if $F(r) = \sum_{k=0}^{\infty} u_k r^k$, then $F(r)/(1 - r)^{N+1} = \sum_{k=0}^{\infty} s_k^{(N)} r^k$. It is easily shown that $s_n^N = \sum_{k=0}^{n} \binom{n-k+N}{N} u_k$, [**22**, p. 96].

If we begin with a vertically convex function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\overline{D}$ and assume that $a_0 = 0$, then we have $f(e^{i\theta}) = u(e^{i\theta}) + iv(e^{i\theta}) = \sum_{n=0}^{\infty} a_n \cos(n\theta) + i\sum_{n=0}^{\infty} a_n \sin(n\theta)$, with $\sum_{n=0}^{\infty} a_n \cos(n\theta)$ monotone decreasing for $0 \le \theta \le \pi$. Our goal is to show that the third mean of f also is vertically convex. It suffices to show that the third mean of $\sum_{n=0}^{\infty} a_n \cos(n\theta)$ is monotone decreasing on $[0, \pi]$. Fejér effectively proves this by showing that the third mean of $-\sum_{n=0}^{\infty} a_n \cos(n\theta) + A$ is monotone increasing, where A is a constant large enough so that $-\sum_{n=0}^{\infty} a_n \cos(n\theta) + A$ is positive on $[0, \pi]$.

The proof is by approximation, and we note that the third mean of the cosine series can be written as an integral convolution of the series with a positive kernel. It is sufficient to prove the theorem for monotone increasing step functions, which uniformly approximate $-\sum_{n=0}^{\infty} a_n \cos(n\theta) + A$; for these functions it is in turn sufficient to prove the theorem for a one-step function of the form

$$g(\theta) = \begin{cases} 0, & 0 \leq \theta \leq a, \quad 0 < a < \pi \\ b, & a < \theta \leq \pi, \end{cases}$$

where $b > 0$. The cosine series for $g(\theta)$ is $2b/\pi\{(\pi - a)/2 - \sum_{k=1}^{\infty} \sin(ka) \cos(k\theta)/k\}$; it is this series whose third mean must be proved increasing. This third mean has

$$s_n^{3\prime}(\theta) = -\sum_{k=1}^{n} \binom{n-k+3}{3} \sin(ka) \sin(k\theta);$$

thus we need to show

$$\sum_{k=1}^{n} \binom{n-k+3}{3} \sin(ka) \sin(k\theta) \geq 0 \quad \text{for } 0 \leq \theta \leq \pi.$$

This occurs, according to a lemma of Fejér's [**3**], if and only if $\sum_{k=1}^{n} \binom{n-k+3}{3} k \sin(k\theta) \geq 0$, for $0 \leq \theta \leq \pi$.

Thus the entire result hinges on non-negativity of the third mean of the series $\sum_{k=0}^{\infty} k \sin(k\theta)$, which Fejér proves by the generating function technique. He proves that all the coefficients in the series for $F(r)/(1-r)^4$ are positive, where

$$F(r) = \sum_{n=0}^{\infty} nr^n \sin(n\theta) = (r \sin\theta) \frac{(1-r^2)}{(1 - 2r\cos\theta + r^2)^2},$$

an identity that follows by differentiating the series form of the Poisson kernel (3). Then he notes that the right hand side equals

$$r \sin(\theta) \frac{1}{1-r^2} \left[\frac{(1-r^2)}{(1-r)^2 (1 - 2r\cos\theta + r^2)} \right]^2.$$

The term $(1-r^2)^{-1}$ clearly has non-negative coefficients in its power series, and the term in the brackets is equal to $\sum_{n=0}^{\infty} [\sin((n+1)\theta/2)/\sin(\theta/2)]^2 r^n$ since it is the Poisson kernel divided by $(1-r)^2$. This series clearly has non-negative coefficients. Thus every factor in the product for $F(r)$ has a power series with non-negative coefficients, and so $F(r)$ itself must have a power series with non-negative coefficients. The theorem in the case in which f is vertically convex in D follows easily.

It is easy to generate examples of univalent polynomials using this theorem. Using the fact that the vertically convex class is closed under convex combinations and that $f_1(z) = z/(1+z)$, $f_2(z) = z/(1-z)$, and $f_3(z) = (i/2)\log((1-iz)/(1+iz))$ are all convex (hence vertically convex), we may form the function $(1/2)f_1(z) + (1/4)f_2(z) + (1/4)f_3(z)$ whose third mean of degree 6 is shown in Figure 3.

As a final note on this theorem we remark that it gave rise to another result of a similar nature due to Pólya and Schoenberg in 1958 [**30**]. In this paper the third Cesàro mean is replaced by the de la Vallée Poussin kernel. It is proved that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is univalent and convex in D, that is, if the image of D under f is a convex set, then the polynomial

$$P_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \omega_n(t - \zeta) f(re^{i\zeta})\, d\zeta, \quad z = re^{it},$$

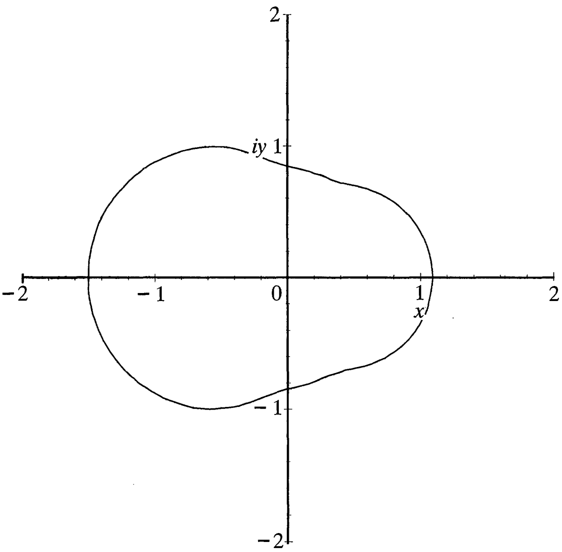


Figure 3. $P(z) = z - \frac{5}{32}z^2 + \frac{5}{21}z^3 - \frac{5}{112}z^4 + \frac{2}{35}z^5 - \frac{1}{224}z^6$.

is convex also, where

$$\omega_n(t) = 1 + 2\sum_{k=1}^{n} \frac{(n!)^2}{(n-k)!(n+k)!}\cos(kt) \geq 0$$

is the de la Vallée Poussin kernel. Thus another positive trigonometric sum gives rise to univalent convex polynomials.

Non-negativity of the third mean of $\sum_{n=1}^{\infty} n\sin(n\theta)$ allowed Fejér to obtain another elementary result, which can also be used to generate examples of vertically convex polynomials **[17]**. We prove the result for polynomials only, though it is valid for Taylor series. Assume that $P(z) = \sum_{n=1}^{N} a_n z^n$ with $a_n \geq 0$. Define $\Delta^1(a_n) = a_n - a_{n-1}$, $\Delta^2(a_n) = \Delta^1(\Delta^1(a_n))$, and so on. Assume now that $\Delta^4(a_n) \geq 0$ for all n, where for $n \geq N$ we put $a_n = 0$; such a sequence is called "fourfold monotone." We show that $P(z)$ is vertically convex: if $P(z) = u(z) + iv(z)$ then $u(e^{i\theta}) = \sum_{n=1}^{N} a_n\cos(n\theta)$, hence $\partial u(e^{i\theta})/\partial\theta = -\sum_{n=1}^{N} a_n(n\sin(n\theta))$. Now summation by parts four times on this sum shows $\partial u(e^{i\theta})/\partial\theta = -\sum_{n=1}^{N} s_n^{(3)}\Delta^4(a_n)$, where $s_n^{(3)}$ is the third order sum of $\sum_{n=1}^{\infty} n\sin(n\theta)$. Thus $\partial u(e^{i\theta})/\partial\theta \leq 0$, and we are done. Fejér was also able to conclude that, under these same hypotheses, $|P(e^{i\theta})|^2$ is a decreasing function of θ for $0 \leq \theta \leq \pi$. Noting that $|P(e^{i\theta})|^2 = A_0 + 2\sum_{n=1}^{\infty} A_n\cos(n\theta)$ with $A_n = \sum_{k=1}^{\infty} a_k a_{k+n}$, he pointed out that fourfold monotonicity of $\{a_k\}$ immediately implies fourfold monotonicity of $\{A_k\}$, whereupon his earlier argument gives the result.

A polynomial satisfying fourfold monotonicity is shown in Figure 4. It is a third Cesàro mean for the function $z/(1-z)$, and there are several points of zero curvature on this curve. Shortly after Fejér proved this theorem, Egervary **[13]** proved that this function is actually convex, not just vertically convex.

Through the years the "fourfold monotone" theorem has been improved: in 1941 Szegö **[38]** replaced fourfold monotonicity by threefold to achieve univalence, but not necessarily vertical convexity, and in 1968 Robertson **[32]** showed that under the hypothesis of threefold monotonicity f is also close-to-convex; see

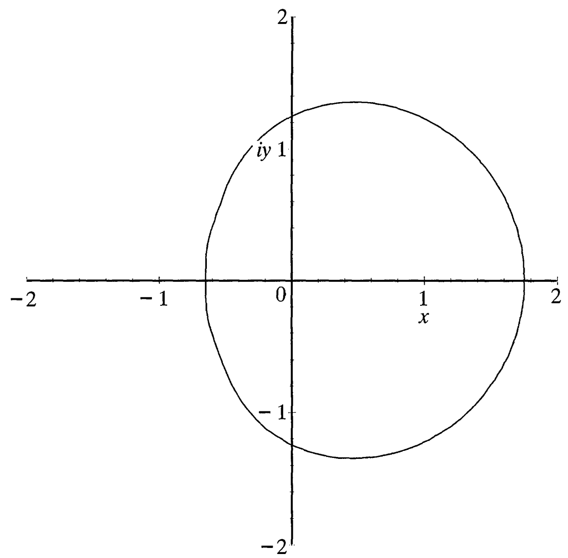


Figure 4. $P(z) = z + \frac{1}{2}z^2 + \frac{1}{5}z^3 + \frac{1}{20}z^4$.

Section 4 for a definition of this property. Convexity of the higher order Cesàro means of $z/(1 - z)$ continues to be of interest [**35**].

4. STARLIKE AND CONVEX CLASSES. Vertical convexity is only one of several kinds of geometric conditions on an analytic function that imply univalence. Better known are starlikeness and convexity: a function analytic in D is said to be *convex* in D if the image of D is a convex set, and it is said to be *starlike* in D if the image of D is convex with respect to the origin. There are analytic conditions that imply these properties: if f is analytic in D, $f(0) = 0$, and $f'(0) = 1$, then f is univalent and starlike in D if and only if $\Re[zf'(z)/f(z)] > 0$ for all $z \in D$; it is univalent and convex in D if and only if $\Re[1 + zf''(z)/f'(z)] > 0$ for all $z \in D$. Furthermore, Alexander proved [**1**] that if f is analytic in D, $f(0) = 0$, and $f'(0) = 1$, then f is convex in D if and only if $zf'(z)$ is starlike; this allows us to derive facts about one class from facts about the other.

We can use these analytic conditions and facts about non-negative trigonometric sums to produce examples of polynomials in the convex and starlike classes. Let $P(z) = \sum_{n=1}^{N} a_n z^n$ with $a_n \geq 0$; assume that $\{na_n\}$ is twofold monotone, where we assume $a_n = 0$, $n \geq N$. Then $\{a_n\}$ is also twofold monotone, and hence if $P(e^{i\theta}) = u(e^{i\theta}) + iv(e^{i\theta}) = \sum_{n=1}^{N} a_n \cos(n\theta) + i\sum_{n=1}^{N} a_n \sin(n\theta)$, we have by a result of Fejér [**18**] that $v(\theta) \geq 0$ for $0 \leq \theta \leq \pi$. Since $v(x) = 0$, $-1 \leq x \leq 1$, it follows by harmonicity of v that $v(re^{i\theta}) > 0$ for $0 < \theta < \pi$, and $0 < r < 1$. Similarly, $v(re^{i\theta}) < 0$ for $\pi < \theta < 2\pi$, $0 < r < 1$. Fejér notes that $\Re[zP'(z)/P(z)] = (uv' - u'v)/(u^2 + v^2)$ and that $2(uv' - u'v) = B_0 + 2\sum_{n=1}^{\infty} B_n \cos(n\theta)$, where $B_n = \sum_{k=1}^{\infty}(2k + n)(a_k a_{k+n})$ $n = 0, 1, 2 \ldots$. He then rewrites $(2k + n)a_k a_{k+n} = ka_k(a_{k+n}) + a_k(k + n)a_{k+n}$ and points out that the twofold monotonicity of $\{ka_k\}$ is passed on to the B_n sequence, whereupon it too produces a non-negative cosine series for the numerator of $\Re[zP'(z)/P(z)]$, $z = e^{i\theta}$. The argument principle forbids any zeroes of P on $\mathbb{R}$ under these circumstances. Thus the numerator series remains non-negative if we move to $re^{i\theta}$, $0 \leq \theta \leq \pi$, $0 < r < 1$, and the

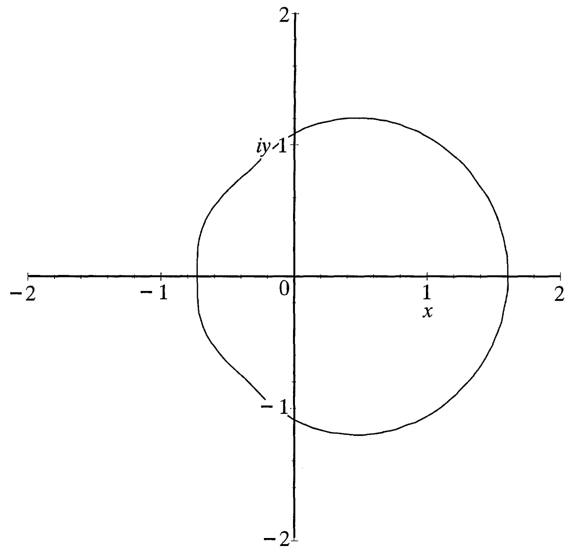


Figure 5. $P(z) = z + \frac{3}{8}z^2 + \frac{1}{6}z^3 + \frac{1}{16}z^4$.

demoninator is nonzero on this circle. Hence, harmonicity of $\Re[zP'(z)/P(z)]$ ensures that it positive throughout rD, $0 < r < 1$. Thus $P(z)$ is starlike univalent in D. See Figure 5 for a simple example.

A different approach to the study of univalent polynomials began in the 1950's. One fixes a degree, say $N = 3$ or 4, and determines the set of all univalent or starlike polynomials $P(z) = \sum_{n=1}^{N} a_n z^n$ of this degree with real coefficients. The problem for univalence and $N = 3$ was solved in several ways ([**10**], [**25**], and [**5**]); the problem for starlikeness and $N = 3$ was solved by Brannan and Brickman in 1975 [**7**]. The solution in these cases produces a set $\{(a_2, a_3)\colon a_2, a_3 \in V_3 \subset \mathbb{R}^2\}$; V_3 is called the "coefficient body" for the problem.

Let us consider the coefficient body of starlikeness for $N = 3$: if we assume $P(z)$ is univalent and starlike in D, and in no larger disc, then $P(z)/z \neq 0$ in $\overline{D}$, and thus $\Re[e^{i\theta}P'(e^{i\theta})/P(e^{i\theta})] \geq 0$, $0 \leq \theta \leq 2\pi$. This is equivalent to $Q(\theta) = (1 + 2a_2^2 + 3a_3^2) + a_2(3 + 5a_3)\cos\theta + 4a_3\cos(2\theta) \geq 0$ for $0 \leq \theta \leq 2\pi$. This in turn requires an easy but messy analysis of conditions on λ_1 and λ_2 that assure $1 + \lambda_1\cos\theta + \lambda_2\cos(2\theta) \geq 0$; the answer is given in [**7**, Lemma 2]. The problem is then to use this lemma to produce the boundary curves for the coefficient body V_3.

Brannan and Brickman obtain the region pictured in Figure 6, whose boundary curves are $a_2^2 = [32a_3(1 - 3a_3)]/(9 - 25a_3)$, $\frac{1}{5} \leq a_3 \leq \frac{1}{3}$, and the lines $a_2 = \pm\frac{1}{2}(1 + 3a_3)$ for $-\frac{1}{3} \leq a_3 \leq \frac{1}{5}$. Thus some elementary facts about lower order trigonometric polynomials can generate the entire coefficient body of third degree starlike polynomials.

In an unpublished result the second author (F. H.) has found the coefficient body $V_4 \subset \mathbb{R}^3$. In a way similar to that in the preceding paragraphs it is easily seen that $P_4(z) = z + a_2z^2 + a_3z^3 + a_4z^4$ starlike in D is equivalent to $Q_4(\theta) = 5a_4\cos(3\theta) + (4a_3 + 6a_2a_4)\cos(2\theta) + (3a_2 + 5a_2a_3 + 7ba_4)\cos\theta + 1 + 2a_2^2 + 3a_3^2 + 4a_4^2 = 20a_4\cos^3\theta + (12a_2a_4 + 8a_3)\cos^2\theta + (7a_3a_4 + 3a_2\ 15a_4 + 5a_2a_3)\cos\theta + 4a_4^2 - 6a_2a_4 + 3a_3^2 - 4a_3 + 1 + 2a_2^2 \geq 0$ for $0 \leq \theta \leq 2\pi$.

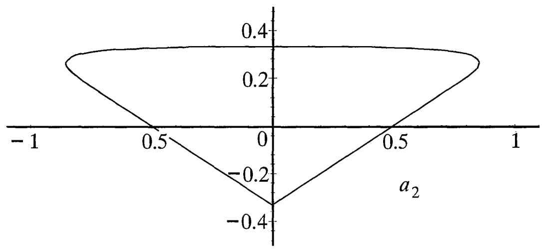
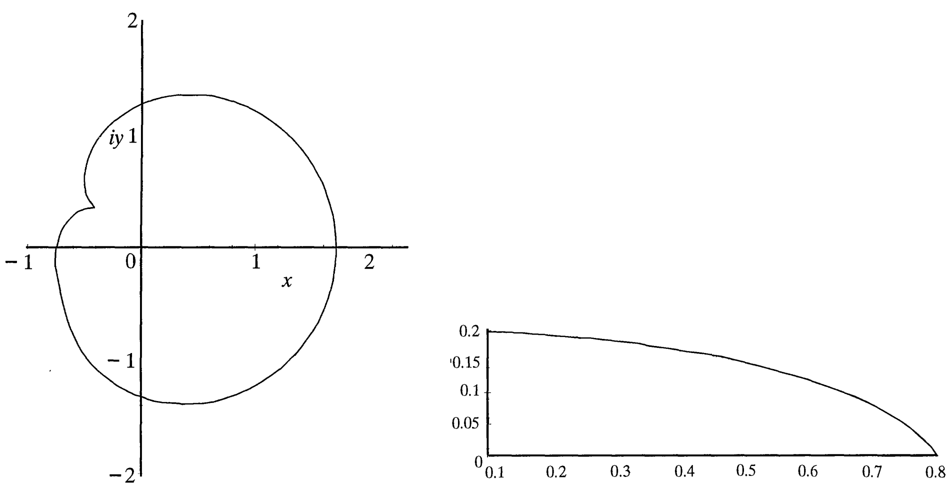


Figure 6. Coefficient body in (a_2, a_3).

This requires a careful analysis of the non-negativity of the cubic $R(x) = x^3 + Bx^2 + Cx + D$ $(x = \cos\theta)$ on $[-1, 1]$. These conditions can be translated to $\mathbb{R}^3$ to find the coefficient body in terms of a_2, a_3, a_4. The resulting coefficient body is found to have a boundary consisting of the planes Π_1: $4a_4 + 3a_3 + 2a_2 + 1 = 0$ and Π_2: $4a_4 - 3a_3 + 2a_2 - 1 = 0$ and a manifold given by a very complicated expression in a_2, a_3, a_4. However, the surfaces can be analyzed and the maximum values of a_2 and a_3 on V_4 can be found $[f_1(z) = z + (1.108\ldots)z^2 + (0.599\ldots)z^3 + (0.1453\ldots)z^4$ maximizes a_2, while $f_2(z) = z + (1.0803\ldots)z^2 + (0.62138\ldots)z^3 + (0.17588\ldots)z^4$ maximizes a_3.] It is interesting to note that although V_3 is a convex body in $\mathbb{R}^2$, V_4 is not convex in $\mathbb{R}^3$; $P_1(z) = z - z^4/4$ and $P_2(z) = z + z^2/2 - 143z^4/1000$ are starlike, but $[P_1(z) + P_2(z)]/2$ is not starlike.

Fejér's result mentioned in the first paragraph of this paper allowed Brannan and Brickman to describe the coefficient body for polynomials of the form $P(z) = z + a_2z^2 + Bz^3$, with $B \in \mathbb{R}$ and $a_2 \in \mathbb{C}$, that are starlike and univalent. This is one of the few cases in which anything is known about polynomials with even one complex coefficient. Brannan and Brickman again analyze $\Re[e^{i\theta}P'(e^{i\theta})\bar{P}(e^{i\theta})] \geq 0$, and produce the relevant non-negative trigonometric sum. The parametrization for the boundary of this region is extremely complicated, although it involves only quadratic and square roots. A Maple V printout of this boundary in the first quadrant for $B = \frac{1}{5}$ is shown in Figure 7, along with a graph



(a) (b)

Figure 7. $P(z) = z + (\frac{1}{2} + i\frac{3}{20})z^2 + \frac{1}{5}z^3$.

of the image of D under $P(z) = z + (10 + 3i)z^2/20 + z^3/5$, a polynomial with coefficients in the region.

In a 1987 paper [**34**] Ruscheweyh used some specific non-negative trigonometric sums to derive other hypotheses that guarantee starlikeness of a polynomial; the sums in question were due to Vietoris; see [**2**, p. 5] for more detailed consideration. Vietoris' inequalities are: if $b_0 > 0$, $b_k \geq 0$, and the b_k's are nonincreasing and satisfy $(2k)b_{2k} \leq (2k-1)b_{2k-1}$, then both $\sum_{k=0}^{n} b_k \cos(k\theta) > 0$ and $\sum_{k=0}^{n} b_k \sin(k\theta) > 0$ for $0 < \theta < \pi$. Ruscheweyh connected two conditions on polynomial coefficients that used these nonnegative sums to prove starlikeness of $f(z) = \sum_{k=1}^{n} a_k z^k$. His conditions are: (A) $a_k \geq 0$, $a_1 = 1$, $(k+1)a_{k+1} \leq k a_k$, for $k \in \mathbb{Z}^+$, and (B) $(2k+1)a_{2k+1} \leq (2k-1)a_{2k}$ for $k \in \mathbb{Z}^+$. He proceeded as follows: if $f(z) = \sum_{k=0}^{n} a_{k+1} z^k$ and $a_{n+1} = 0$, then $f'(z) = \sum_{k=0}^{n} (k+1)a_{k+1} z^k$. With $b_k \equiv (k+1)a_{k+1}$ we see that (A) and (B) imply (A′): $\sum_{k=0}^{n}(k+1)a_{k+1}\cos(k\theta) > 0$, and (B′): $\sum_{k=0}^{n}(k+1)a_{k+1}\sin(k\theta) > 0$ for $0 < \theta < \pi$ by Vietoris' inequalities. Then (A′) and harmonicity of $\Re[f'(z)]$ imply (A″): $\Re[f'(z)] > 0$ on D, and harmonicity of $\Im[f'(z)]$ on $D \cap \{z : \Im(z) > 0\}$ implies either $f(z) = z$ or $\Im[f'(z)] > 0$ in $D \cap \{z : \Im(z) > 0\}$. The condition $\Im(f'(z)) > 0$ in $D \cap \{z : \Im(z) > 0\}$ is equivalent to (B″): $\Im[f'(z)]\Im(z) > 0$ on $z \in D \setminus \mathbb{R}$.

The two conditions (A″) and (B″) together imply starlikeness. We use $f(z)/(zf'(z))$ rather than its reciprocal, and note that

$$\frac{f(z)}{zf'(z)} = \int_0^1 \frac{f'(tz)}{f'(z)}\, dt, \quad z \in D.$$

Now if $\Im(z) > 0$, then (B″) implies $\Im[f'(tz)] > 0$ and $\Im[f'(z)] > 0$, and (A″) implies $\Re[f'(tz)] > 0$ and $\Im[f'(z)] > 0$. Since the numerator and denominator of the integrand are in the first quadrant, their quotient has positive real part for each t. It follows that $\Re[f(z)/zf'(z)] > 0$ in $\Im(z) > 0$; a similar argument works for $\Im(z) < 0$. Thus f is starlike in D. An example of a polynomial satisfying this condition is given in Figure 8.

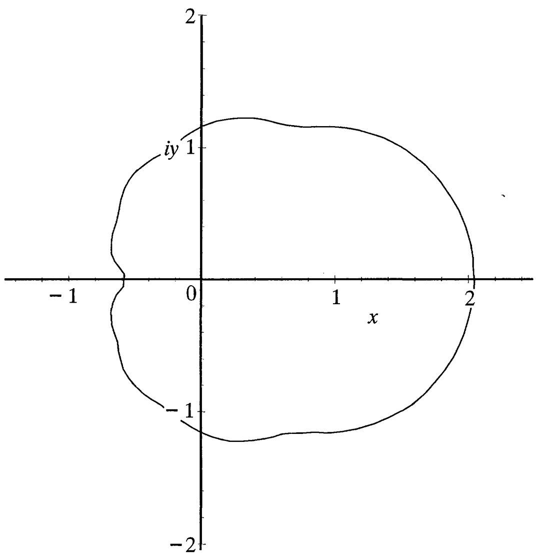


Figure 8. $P(z) = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{8}z^4 + \frac{3}{40}z^5 + \frac{1}{16}z^6 + \frac{5}{112}z^7 + \frac{5}{128}z^8 + \frac{35}{112}z^9$.

It is worth noting that condition (A″) alone implies univalence; this is known as the Noshiro-Warshawski Theorem, and it follows immediately from the fundamental theorem of calculus: if $z_1 \neq z_2 \in D$, then

$$f(z_1) - f(z_2) = -\int_{z_1}^{z_2} f'(z)\,dz = (z_2 - z_1)\int_0^1 f'(tz_2 + (1-t)z_1)\,dt \neq 0$$

by (A″). A function satisfying (A″) is an example of a *close-to-convex function*: for increasing θ the argument of the tangent vector to $f(re^{i\theta})$ never decreases by more than π from any preceding value. Using the condition (A″) and any non-negative trigonometric sum it is easy to construct examples of close-to-convex polynomials.

5. COEFFICIENT BOUNDS. The univalent polynomial $P_N(z) = \sum_{n=1}^N a_n z^n$ and the non-negative trigonometric sum $T_N(\theta) = \lambda_0 + \sum_{n=1}^N \lambda_n \cos(n\theta) + \mu_n \sin(n\theta)$ have a common feature: there are bounds on the sizes of their coefficients. In the case of $T_N(\theta) \geq 0$ one can obtain a trivial bound by noting that

$$\begin{aligned} |\lambda_n| &= \left| \frac{1}{\pi}\int_{-\pi}^{\pi} T_N(\theta)\cos(n\theta)\,d\theta \right| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |T_N(\theta)||\cos(n\theta)|d\theta \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} T_N(\theta)|\cos(n\theta)|\,d\theta \leq \frac{1}{\pi}\int_{-\pi}^{\pi} T_N(\theta)d\theta = 2\lambda_0. \end{aligned} \tag{5}$$

These bounds can be improved: with $\lambda_0 = 1$, Fejér [31] obtained $\sqrt{\lambda_1^2 + \mu_1^2} \leq 2\cos(\pi/(n+2))$ and $|\lambda_N| \leq 1$. Holland [23] obtained bounds for all λ_n, provided $\mu_n = 0$ for all n, that is, $|\lambda_n| \leq 2\cos(\pi/(\nu+2))$, where ν is the greatest integer less than or equal to N/n. His proof involves much computation but uses only basic facts about linear algebra and complex numbers.

For univalent polynomials the only easy bound is on a_N; since $P_N'(z) \neq 0$ in D it follows easily that $|a_N| \leq 1/N$. For the remaining coefficients we can obtain bounds if the a_n are real. Dieudonné did this in 1931 [11] by using his criterion (4). He noted that univalence of $P_N(z)$ implies $D(x,\phi) = \sum_{n=1}^N a_n x^{n-1}\sin(n\phi)/\sin\phi \geq 0$ for $-1 \leq x \leq 1$, since $a_1 = 1 > 0$. He then constructed

$$\begin{aligned} \psi(x,\phi) &= 2\sin^2\phi D(x,\phi) \\ &= 1 + a_2 x\cos\phi + (a_3x^2 - 1)\cos(2\phi) + \cdots \\ &\qquad + (a_n x^2 - a_{n-2}x^{n-3})\cos((n-1)\phi) + \cdots + a_n x^{n-1}\cos((n+1)\phi). \end{aligned}$$

Thus for any fixed $x \in [-1,1]$, $\psi(x,\phi)$ is a non-negative cosine sum. Applying the bound (5) and letting $x \to 1$ yields the bounds $|a_2| \leq 2$, $|a_3 - 1| \leq 2$, and $|a_n - a_{n-2}| \leq 2$ for $n \geq 4$. It follows easily by induction that $|a_n| \leq n$ for all n, which verifies the Bieberbach conjecture for univalent polynomials with real coefficients. These bounds can be improved by using the sharper estimates for $|\lambda_n|$ obtained by Fejér and Holland [23].

In a 1970 paper Michel [29] proceeded along similar lines to obtain a sharp bound on coefficients of univalent polynomials $P_4(z) = \sum_{n=1}^4 a_n z^n$ with real coefficients and $|a_4| \leq 1/4$. Here is his argument for $|a_2|$. Working with $D_4(x,\phi) = \sum_{n=1}^4 a_n x^{n-1}\sin(n\phi)/\sin\phi \geq 0$ he let $x \to 1$ and rewrote $D_4(x,\phi)$ itself as a cosine series to obtain $(1 + a_3) + 2(a_2 + a_4)\cos\phi + 2a_3\cos(2\phi) + 2a_4\cos(3\phi) \geq 0$ for $0 \leq \phi \leq 2\pi$. He then used this to obtain two inequalities: by using Fejér's bound on $|\lambda_1|$ he gets $(a_2 + a_4) \leq (1/4)(1+\sqrt{5})(1+a_3)$, and making the clever choice of $\phi = 3\pi/5$ he gets $(a_2 + a_3) \leq (1/2)(1+\sqrt{5})(1+a_4)$. These two inequalities can be combined with the bound $|a_4| \leq 1/4$ to obtain

$|a_2| \le (3/8)(1+\sqrt{5}) = 1.2135\ldots$. This bound was obtained earlier by Suffridge **[37]** as a special case of his results on univalent polynomials with real coefficients and $a_N = 1/N$. He also showed that the polynomial $P_4(z) = z + (3/8)(1+\sqrt{5})z^2 + (1/4)(1+\sqrt{5})z^3 + (1/4)z^4$ is univalent. We conclude by turning to Suffridge's results in a different context.

6. NON-NEGATIVE TRIGONOMETRIC SUMS AS REPRESENTING MEASURES. Here is how the first author (A. G.) became interested in these matters: In **[37]** Suffridge introduced the polynomials $P(z,n,j) = \sum_{k=1}^{n} A_{k,j} z^k$, where

$$A_{k,j} = \frac{n-k+1}{n}\frac{\sin\left(k\alpha_{j,n}\right)}{\sin\alpha_{j,n}}, \alpha_{j,n} = \frac{j\pi}{n+1}, \quad j,k = 1,\ldots,n; \tag{6}$$

note that $A_{n,j} = +1/n$, for all j. For $a_1 = 1$, a_k real for $k = 2,\ldots,n$, and $|a_n| = 1/n$ he proved that if $f(z) = \sum_{k=1}^{n} a_k z^k$ is univalent in D (the class of such polynomials is denoted by Q), then $f(z) = \sum_{m=1}^{n} \alpha_m P(z,n,m)$ for some $\alpha_m \ge 0$, $m = 1,2,\ldots,n$ and $\sum_{m=1}^{n} \alpha_m = 1$. These polynomials have other extremal properties including those involving the coefficients mentioned in the last section: $|a_k| \le A_{k,1}$, for instance, if $a_n = 1/n$. In independent work MacGregor, Brickman, Hallenbeck, and Wilken (**[8]**, **[9]**, **[26]**; see also **[27]**) considered classes of functions $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ analytic in D for which

$$f(z) = \int_0^{2\pi} \frac{z}{\left(1 - e^{i\theta}z\right)^{\alpha}}\, d\mu(\theta) \quad \text{for } \alpha > 0, \tag{7}$$

for some probability measure μ on $[0, 2\pi]$. The class of such f is denoted $\mathscr{F}_\alpha^0$; the class with

$$f(z) = \int_0^{2\pi} \frac{1}{\left(1 - e^{i\theta}z\right)^{\alpha}}\, d\mu(\theta)$$

is called $\mathscr{F}_\alpha$. They prove that $\mathscr{F}_2^0$ can be described as the closed convex hull of the family of starlike functions in the topology of uniform convergence on compact subsets; in particular, all starlike mappings are in $\mathscr{F}_2^0$. They pointed out that all univalent functions with real coefficients are also in $\mathscr{F}_2^0$, although MacGregor **[26]** constructed a univalent function in D for which there is no complex Borel measure giving the representation (7) for $\alpha = 2$.

The following question occurred to the first author. If $P(z,n,j) \in \mathscr{F}_2^0$ for all n, j, then what measure $\mu = \mu_{n,j}$ corresponds to $P(z,n,j)$ in the representation (7)? An easy calculation shows that

$$d\mu_{n,j} = \frac{1}{\pi}\left[\frac{1}{2} + \sum_{k=2}^{n} \frac{n-k+1}{n}\frac{\sin\left(k\alpha_{j,n}\right)}{k\sin\alpha_{j,n}}\cos\left((k-1)\theta\right)\right] d\theta;$$

it follows that the bracketed trigonometric sum must be non-negative. Is there an alternative way to prove the non-negativity of this sum?

We are indebted to Richard Askey for pointing us toward **[3]**, where elementary means are used to prove that

$$F(r,\alpha,\theta) \equiv \frac{1}{2} + \sum_{k=1}^{\infty} r^k \frac{\sin\left((k+1)\alpha\right)}{(k+1)\sin\alpha}\cos(k\theta) \ge 0,$$
$$-1 < r < 1,\ 0 \le \alpha,\ \theta \le \pi;$$

the positivity of $\mu_{n,j}$ then follows by convolving $F(r, \alpha, \theta)$ with the Fejér kernel and letting $r \to 1$. Moreover, the reasoning can be reversed: $F(r, \alpha, \theta) \geq 0$ implies that $\mu_{n,j} \geq 0$ for all n and j; thus $P(z, n, j) \in \mathscr{F}_2^0$ for all n and j. It follows from the representation of an arbitrary $f \in Q$ as a convex combination of $P(z, n, j)$ that $Q \subset \mathscr{F}_2^0$. Furthermore, denseness of the class Q in the set of all univalent f with real coefficients [36] and compactness of $\mathscr{F}_2^0$ shows that all univalent functions with real coefficients are in $\mathscr{F}_2^0$. Thus inclusion of these univalent functions in $\mathscr{F}_2^0$ follows from the non-negativity of $F(r, \alpha, \theta)$.

There is more going on here: the proof in [9] that univalent functions with real coefficients are in $\mathscr{F}_2^0$ followed from the observation that such a function is *typically real*, i.e., it satisfies $f(re^{i\theta})\sin\theta > 0$ for all r, θ, and that all typically real functions are in $\mathscr{F}_2^0$. This follows in turn from a representation theorem due to Robertson stating that if f is typically real then

$$f(z) = \int_0^{\pi} \frac{z}{(1 - e^{it}z)(1 - e^{-it}z)} d\mu(t)$$

for some probability measure μ. Brickman, MacGregor, and Wilken note that to prove the inclusion it is sufficient to prove that $(1 - e^{it}z)^{-1}(1 - e^{-it}z)^{-1} \in \mathscr{F}_2$, for each $t \in [0, \pi]$, and this is a special case of a much more general theorem [9, p. 96]. A first glance at the proof of this theorem doesn't yield the representing measure μ_t for $(1 - e^{it}z)^{-1}(1 - e^{-it}z)^{-1}$, but it follows easily that

$$\begin{aligned} \int_0^{2\pi} \frac{z}{(1 - e^{i\theta}z)^2} F(r, t, \theta)\, d\theta &= \frac{1}{r}\left[\sum_{k=1}^{\infty} (rz)^k \frac{\sin(kt)}{\sin t}\right] \\ &= \frac{1}{r}\left[\frac{rz}{(1 - re^{it}z)(1 - re^{-it}z)}\right] \end{aligned} \tag{8}$$

so that

$$(1 - e^{it}z)^{-1}(1 - e^{-it}z)^{-1} = \lim_{r \to 1} \int_0^{2\pi} \frac{z}{(1 - e^{i\theta}z)^2} F(r, t, \theta)\, d\theta.$$

The representing measure for $(1 - e^{it}z)^{-1}(1 - e^{-it}z)^{-1}$ is $\lim_{r \to 1} F(r, t, \theta)$ in the weak star topology; for example, if $t = 0$ it is the point mass at 0. The Suffridge polynomials (6) are "smoothed out" partial sums of $z/(1 - e^{it}z)(1 - e^{-it}z)$ for certain choices of t, and their representing measures are "smoothed-out" versions of $F(r, t, \theta)$ for those same choices of t, as $r \to 1$.

Let us conclude by returning to the vertically convex class. One can show that the vertically convex class is a subset of $\mathscr{F}_1^0$ as follows: f is vertically convex if and only if zf' is typically real [20, p. 206]; if zf' is typically real, then $zf' \in \mathscr{F}_2^0$ [9, p. 96]; finally if $zf' \in \mathscr{F}_2^0$, then $f \in \mathscr{F}_1^0$, the closed convex hull of the convex functions [27, p. 399]. Thus there are many non-convex univalent functions in $\mathscr{F}_1^0$, and more can be concocted by finding representing measures relative to $\mathscr{F}_1^0$ for some univalent polynomials. We conclude with such an example, the polynomials of Alexander [1] considered at the beginning of Section 2. The polynomial $P_n(z) = \sum_{k=1}^{n} z^k/k$ is univalent and Brannan [6] proved it to be close-to-convex, but even $z + z^2/2 + z^3/3$ is outside the body of starlikeness given in Figure 7. We can represent

$$P_n(z) = \int_0^{2\pi} \frac{z}{(1 - e^{i\theta}z)} d\mu_n(\theta)$$

for $\mu_n(\theta) = 1/2 + \cos\theta/2 + \cos(2\theta)/3 + \cdots + \cos((n-1)\theta)/n$. That $\mu_n(\theta) \geq 0$ is due to Rogosinski-Szegö **[33]**, so this non-negative trigonometric sum gives an easy example of an $f \in \mathscr{F}_1^0$ that fails even to be starlike, let alone convex.

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From the MONTHLY 50 years ago...

Professor Z. T. Gallion of Southwestern Louisiana Institute presided at a joint banquet with the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics that was held at Oak Grove Inn Friday evening. Professor C. V. Newsom of Oberlin College, Oberlin, Ohio, was guest speaker for both organizations at the banquet and again on Saturday morning. His addresses were:

1. *Relationship of the Association and the National Council.*
After reviewing recent developments in the field of mathematical education, Mr. Newsom emphasized that many urgent problems could be solved only by the cooperation of college and secondary teachers. In particular, he suggested a study of the entire mathematics curriculum from the first grade to graduate school in the light of new mathematical knowledge that is available, the needs of modern science, and the teaching problems introduced as a result of mass education.

2. *Mathematics and our culture.*
This paper presented some points of view in regard to the significance of mathematics in man's attempt to comprehend his environment. Mr. Newsom expressed the idea that more of the philosophical contributions of mathematics should be a part of all elementary courses in mathematics.

...from a report of a meeting of the Louisiana-Mississippi Section
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