PACIFIC JOURNAL OF MATHEMATICS
Vol. 74, No. 2, 1978

SEQUENCES OF BOUNDED SUMMABILITY DOMAINS

R. M. DEVOS AND F. W. HARTMANN

C. Goffman and G. N. Wollan conjectured that the bounded summability field of a regular matrix A is so thin that the union of countably many such sets is not dense in m. G. M. Petersen proved this conjecture. This result is strengthened by showing if A is a noncoercive matrix whose summability field contains all the finite sequences then its bounded summability field is so thin that the union of countably many such sets is not dense in m. An example is given to show that the condition of containing the finite sequences is necessary.

Preliminaries. Let m and c be respectively the Banach spaces of bounded and convergent sequences, $x = \{x_n\}$, of complex numbers with norm $\|x\|_\infty = \sup_n |x_n|$, $B(x, r) = \{z \in m: \|x + z\|_\infty < r\}$. Denote the nth section of x by $P_n(x) = (x_1, \cdots, x_n, 0, 0, \cdots)$. For each infinite matrix A the set of x transformed by A to convergent sequences is called the summability field of A and denoted by c_A. The set of bounded sequences in c_A is called the bounded summability field of A and is denoted by $\mathscr{A}$. A is called conservative if and only if $c_A \supset c$, regular if and only if A is conservative and limits are preserved, coercive if and only if $c_A \supset m$. If $A = (a_{nk})$, then the A transform of x is designated by $Ax = \{(Ax)_n\} = \{\sum_k a_{nk}x_k\}$. A is conservative if and only if $\|A\|_\infty = \sup_n \sum_k |a_{nk}| < \infty$, $a_k = \lim_n a_{nk}$ exists for each k and $\lim_n \sum_k a_{nk}$ exists [5, p. 165]. A is coercive if and only if $\sum_k |a_{nk}|$ converges uniformly in n and a_k exists for each k [5, p. 169]. Define the essential norm of A by $\|A\|_e = \limsup_n \sum_k |a_{nk} - a_k|$ whenever a_k exists for each k. (Note $\|\ \|_e$ is not a true norm, since $\|\ \|_e$ may be infinite.)

Let E^∞ be the set of all finite sequences and N_0 the set of all sequences of 0's and 1's. Using binary expansions there is a natural injective mapping of $(0, 1)$ onto all but a countable subset of N_0.

MAIN RESULTS. C. Goffman and G. N. Wollan conjectured [4] that the bounded summability field of regular A is so thin that the union of countably many such sets is not dense in m. G. M. Petersen proved this conjecture [6]. We strengthen that result and show that in a certain sense our result is best possible.

THEOREM. *Let $\{A_i\}$ be a countable collection of noncoercive matrices with $\mathscr{A}_i \supset E^\infty$, $i = 1, 2, \cdots$, then $\bigcup_{i=1}^\infty \mathscr{A}_i$ is not dense in m.*

We prove the theorem through a series of lemmas. Since we want $E^\infty \subset \mathscr{A}$, we shall assume all A in the sequel have convergent columns.

LEMMA 1. *Let $\|A\|_\infty < \infty$ then $\|A\|_c = 0$ if and only if A is coercive.*

Proof. Suppose A is coercive. Let $\varepsilon > 0$. There exists k_0

$$\sum_{k=k_0+1}^{\infty} |a_{nk}| < \varepsilon/3$$

for all n. Since $\{a_k\} \in \ell^1$, there is a k_1 such that $k > k_1$ implies

$$\sum_{k=k_1+1}^{\infty} |a_k| < \varepsilon/3 .$$

Let $k_2 = \max(k_1, k_0)$. There exists $n_0 = n_0(k_2)$ such that $n > n_0$ implies

$$\sum_{\kappa=1}^{k_2} |a_{nk} - a_k| < \varepsilon/3 .$$

Let $n > n_0$ then

$$\begin{aligned}\sum_{k=1}^{\infty} |a_{nk} - a_k| &= \sum_{k=1}^{k_2} |a_{nk} - a_k| + \sum_{k=\kappa_2+1}^{\infty} |a_{nk} - a_k| \\ &\leq \sum_{k=1}^{k_2} |a_{nk} - a_k| + \sum_{k=k_2+1}^{\infty} |a_{nk}| + \sum_{k=k_2+1}^{\infty} |a_k| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .\end{aligned}$$

Conversely assume A is noncoercive. There exists $\varepsilon > 0$ and an increasing sequence of positive integers $\{n(p)\}_{p=1}^{\infty}$ such that $\sum_{k=p+1}^{\infty} |a_{n(p),k}| > \varepsilon$. There exists k_0 such that $\sum_{k=k_0+1}^{\infty} |a_k| < \varepsilon/2$. Pick p with $p > k_0$ then

$$\begin{aligned}\sum_{k=1}^{\infty} |a_{n(p),k} - a_k| &\geqq \sum_{k=k_0+1}^{\infty} |a_{n(p),k} - a_k| \\ &\geqq \sum_{k=k_0+1}^{\infty} |a_{n(p),k}| - \sum_{k=k_0+1}^{\infty} |a_k| \\ &\geqq \varepsilon - \varepsilon/2 = \varepsilon/2 .\end{aligned}$$

Therefore $\|A\|_c > 0$.

Let $\Gamma(c, c)$ be the Banach algebra of conservative matrices and $\mathscr{K}$ be the ideal of compact operators. It is well known [8] that $A \in \mathscr{K}$ if and only if A is coercive. $\Gamma(c, c)/\mathscr{K}$ is a Banach algebra and is called a Calkin algebra [2]. It is easily seen that $\| \ \|_c$ is the norm in the Calkin algebra.

LEMMA 2. *Let $\|A\|_c < \infty$ and a and b be cluster points of Ax,* want $E^\infty \subset \mathscr{A}$, we shall assume all A in the sequel have convergent columns.

LEMMA 1. *Let* $\|A\|_\infty < \infty$ *then* $\|A\|_c = 0$ *if and only if* A *is coercive.*

Proof. Suppose A is coercive. Let $\varepsilon > 0$. There exists k_0

$$\sum_{k=k_0+1}^{\infty} |a_{nk}| < \varepsilon/3$$

for all n. Since $\{a_k\} \in \ell^1$, there is a k_1 such that $k > k_1$ implies

$$\sum_{k=k_1+1}^{\infty} |a_k| < \varepsilon/3 .$$

Let $k_2 = \max(k_1, k_0)$. There exists $n_0 = n_0(k_2)$ such that $n > n_0$ implies

$$\sum_{\kappa=1}^{k_2} |a_{nk} - a_k| < \varepsilon/3 .$$

Let $n > n_0$ then

$$\begin{aligned}\sum_{k=1}^{\infty} |a_{nk} - a_k| &= \sum_{k=1}^{k_2} |a_{nk} - a_k| + \sum_{k=\kappa_2+1}^{\infty} |a_{nk} - a_k| \\ &\leq \sum_{k=1}^{k_2} |a_{nk} - a_k| + \sum_{k=k_2+1}^{\infty} |a_{nk}| + \sum_{k=k_2+1}^{\infty} |a_k| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .\end{aligned}$$

Conversely assume A is noncoercive. There exists $\varepsilon > 0$ and an increasing sequence of positive integers $\{n(p)\}_{p=1}^{\infty}$ such that $\sum_{k=p+1}^{\infty} |a_{n(p),k}| > \varepsilon$. There exists k_0 such that $\sum_{k=k_0+1}^{\infty} |a_k| < \varepsilon/2$. Pick p with $p > k_0$ then

$$\begin{aligned}\sum_{k=1}^{\infty} |a_{n(p),k} - a_k| &\geq \sum_{k=k_0+1}^{\infty} |a_{n(p),k} - a_k| \\ &\geq \sum_{k=k_0+1}^{\infty} |a_{n(p),k}| - \sum_{k=k_0+1}^{\infty} |a_k| \\ &\geq \varepsilon - \varepsilon/2 = \varepsilon/2 .\end{aligned}$$

Therefore $\|A\|_c > 0$.

Let $\Gamma(c, c)$ be the Banach algebra of conservative matrices and $\mathscr{K}$ be the ideal of compact operators. It is well known [8] that $A \in \mathscr{K}$ if and only if A is coercive. $\Gamma(c, c)/\mathscr{K}$ is a Banach algebra and is called a Calkin algebra [2]. It is easily seen that $\|\ \|_c$ is the norm in the Calkin algebra.

LEMMA 2. *Let* $\|A\|_c < \infty$ *and* a *and* b *be cluster points of* Ax,

$x \in m$, then $|a - b| \leqq 2\|A\|_c \|x\|_\infty$.

Proof. Let a and b be cluster points of Ax and $\varepsilon > 0$. There exist increasing sequences of positive integers $\{n(i)\}$, $\{m(j)\}$ and N_0 such that for $n(i), m(j) > N_0$

$$\left|\sum_k a_{n(i),k} x_k - a\right| < \varepsilon$$

and

$$\left|\sum_k a_{m(j),k} x_k - b\right| < \varepsilon .$$

There exists N_1 such that $n > N_1$ implies

$$\sum_k |a_{nk} - a_k| < \|A\|_c + \varepsilon .$$

Let $n(i), m(j) > \max(N_0, N_1)$ then

$$\begin{aligned} |a - b| &\leqq \left|\sum_k a_{n(i),k} x_k - \sum_k a_{m(j),k} x_k\right| + 2\varepsilon \\ &\leqq \sum_k |a_{n(i),k} - a_{m(j),k}|\,|x_k| + 2\varepsilon \\ &\leqq \|x\|_\infty \sum_k |(a_{n(i),k} - a_k) - (a_{m(j),k} - a_k)| + 2\varepsilon \\ &\leqq \|x\|_\infty(\|A\|_c + \varepsilon + \|A\|_c + \varepsilon) + 2\varepsilon . \end{aligned}$$

Since ε is arbitrary the conclusion follows.

The next lemma is due to Bennett and Kalton and appears as Lemma 7 of [1, p. 577].

LEMMA 3. (*Bennett and Kalton*). *If $z_1, z_2, \cdots, z_n$ is any finite collection of complex numbers then there exists a subset $J(n)$ of $\{1, \cdots, n\}$ such that*

$$\left|\sum_{j \in J(n)} z_j\right| \geqq \frac{1}{4} \sum_{i=1}^{n} |z_i| .$$

LEMMA 4. *If $\|A\| = \infty$, then there exists $E(A)$ with $E(A) \subset N_0$, $N_0 \backslash E(A)$ of first category and if $u \in E(A)$ then $B(u, 1/32) \cap \mathscr{A} = \varnothing$.*

Proof. Case 1. Assume all the rows of A are in ℓ^1. Let $\|A\| = \infty$. Pick sequences $n(k)$ and $q(k)$ inductively such that $n(1) = 1$ and

(i) $\sum_{i=q(k)+1}^{\infty} |a_{n(k),i}| < 2^{-k}$

(ii) $\sum_{i=q(k-1)+1}^{q(k)} |a_{n(k),i}| > (65/7) \sup_j \{\sum_{i=1}^{q(k-1)} |a_{ji}|\}$.

By Lemma 3 select $J(k) \subset \{q(k-1) + 1, \cdots, q(k)\}$ with

$$\left|\sum_{i\in J(k)} a_{n(k),i}\right| \geqq \frac{1}{4}\sum_{i=q(k-1)+1}^{q(k)} |a_{n(k),i}| \,.$$

For each natural number k define the sequence u^k by $u_i^k = 1$ if $i \in J(k)$, $u_i^k = 0$ if $i \notin J(k)$. Let

$$O_k = \{u \in N_0 \colon (P_{q(k)} - P_{q(k-1)})(u - u^k) = 0\} \,.$$

If $E(A) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} O_k$, then $E(A)$ is of second category. [$\bigcup_{k=n}^{\infty} O_k$ is open and dense, hence by the Baire theorem $E(A)$ is of second category.] Let $u \in E(A)$ and $\|z\|_\infty < 1/32$. u is in an infinite number of the O_k. Let $u \in O_r$. Then

$$\begin{aligned}
|(A(u+z))_{n(r)}| &\geqq |(Au)_{n(r)}| - |(Az)_{n(r)}| \\
&\geqq \left|\sum_{i=q(r-1)+1}^{q(r)} a_{n(r),i}u_i\right| - \left|\sum_{i=1}^{q(r-1)} a_{n(r),i}u_i\right| - \left|\sum_{i=q(r)+1}^{\infty} a_{n(r),i}u_i\right| \\
&\quad - \frac{1}{32}\sum_{i=1}^{\infty} |a_{n(r),i}| \\
&\geqq \frac{1}{4}\sum_{i=q(r-1)+1}^{q(r)} |a_{n(r),i}| - \frac{33}{32}\sum_{i=1}^{q(r-1)} |a_{n(r),i}| \\
&\quad - \frac{33}{32}\sum_{i=q(r)+1}^{\infty} |a_{n(r),i}| - \frac{1}{32}\sum_{i=q(r-1)+1}^{q(r)} |a_{n(r),i}| \\
&\geqq \frac{7}{32}\sum_{i=q(r-1)+1}^{q(r)} |a_{n(r),i}| - \frac{33}{32}\sum_{i=1}^{q(r-1)} |a_{n(r),i}| - \frac{33}{32}2^{-r} \\
&\geqq \frac{7}{32}\,\frac{65}{7}\sup_j \left\{\sum_{i=1}^{q(r-1)} |a_{ji}|\right\} - \frac{33}{32}\sup_j \left\{\sum_{i=1}^{q(r-1)} |a_{ji}|\right\} - 2^{1-r} \\
&\geqq \sup_j \left\{\sum_{i=1}^{q(r-1)} |a_{ji}|\right\} - 2^{1-r} \longrightarrow \infty \quad \text{as} \quad r \longrightarrow \infty \,.
\end{aligned}$$

Hence the A transform of $u + z$ is unbounded.

Case 2. Let A have one row, x, not in ℓ^1. Let $B = (b_{nk})$ where $b_{nk} = P_n(x)$, $n = 1, 2, \cdots$. Then $\mathscr{A} \subset \mathscr{B}$ and B satisfies the hypothesis of Case 1. Let $E(A) = E(B)$ then $E(A) \cap \mathscr{A} = \varnothing$ and $E(A)$ satisfies the other conditions of the lemma's conclusion.

LEMMA 5. *If $\|A\| < \infty$, and A is noncoercive then there is $E(A)$ with $E(A) \subseteq N_0$, $N_0 \backslash E(A)$ is of first category and if $u \in E(A)$, then $B(u, 1/32) \cap \mathscr{A} = \varnothing$.*

Proof. Case 1. Assume $a_k = 0$, $k = 1, 2, \cdots$. Let α^n be the nth row of A. Using an argument similar to that of Petersen and Baker [6] (see also the construction of Lemma 4) it can be shown that without lose of generality one may assume that the rows and columns of A are in E^∞ and moving to the right, (if $P_j\alpha^n = 0$ then

$P_j\alpha^m = 0$ for $m \geqq n$). By Lemma 1 $\|A\|_c > 0$. Hence there exists increasing sequences $n(j)$ and $r(j)$ of positive integers such that

(i) $\sum_{k=r(j-1)+1}^{r(j)} |a_{n(j),k}| > \|A\|_c/2$

(ii) $(P_{r(j)} - P_{r(j-1)})\alpha^{n(j)} = \alpha^{n(j)}$.

Let $J(2j)$ be a subset of $r(2j-1)$ to $r(2j)-1$ with

$$\left|\sum_{k\in J(2j)} a_{n(j),k}\right| \geqq \frac{1}{4}\sum_{j=r(2j-1)+1}^{r(2j)} |a_{n(j),k}| \geqq \frac{\|A\|_c}{8}$$

(see Lemma 3). Define $O_j = \{u \in N_0 \colon u_k = 1$ if $k \in J(2j)$, $u_k = 0$ if $r(2j-2)+1 \leqq k \leqq r(2j)$, $k \notin J(2j)\}$. Since only a finite number of coordinates are specified for elements of O_j, O_j is open. For each k, $\bigcup_{j=k}^{\infty} O_j$ is open and dense, hence by the Baire category theorem. $\bigcap_{k=1}^{\infty}\bigcup_{j=k}^{\infty} O_j$ is of second category. Let $E(A) = \{u \in N_0 \colon Au$ has cluster points, a, b, with $|a-b| \geqq \|A\|_c/8\}$. By construction each element of $\bigcap_{k=1}^{\infty}\bigcup_{j=k}^{\infty} O_j$ has 0 and a ($|a| > \|A\|_c/8$) as cluster points thus $E(A)$ is of second category. Let $u \in E(A)$ and $\|z\|_\infty < 1/32$ and consider $A(u+z)$. Au has two cluster points separated in distance by at least $\|A\|_c/8$, and $A(z)$ has cluster points separated by at most $2(1/32)\|A\|_c$ (Lemma 2). Therefore $A(u+z)$ has at least two cluster points; hence $u + z \notin \mathscr{A}$.

Case 2. Let $a_k \neq 0$ for some k. Define $B = (b_{nk})$ where $b_{nk} = a_k$, $n, k = 1, 2, \cdots$. B transforms every bounded sequence to a constant sequence, thus the cluster points of $(A-B)u$, $u \in m$, are a shift of those of Au, and $A - B$ satisfies the hypothesis of Case 1. Thus the conclusion follows in a manner similar to Case 1.

Proof of Theorem. Let A_i be a countable collection of non-coercive matrices with $\mathscr{A}_i \supset E^\infty$, $i = 1, 2, \cdots$. By Lemmas 4 and 5 for each i there exists $E(A_i) \subseteqq N_0$, $E(A_i)$ of second category, and if $u \in E(A_i)$, $B(u, 1/32) \cap \mathscr{A}_i = \varnothing$. Thus $\bigcap_{i=1}^{\infty} E(A_i) \neq \varnothing$ and if $u \in \bigcap_{i=1}^{\infty} E(A_i)$, then $B(u, 1/32) \cap (\bigcup_{i=1}^{\infty} \mathscr{A}_i) = \varnothing$. Hence $\bigcup_{i=1}^{\infty} \mathscr{A}_i$ is not dense in m.

Goffman and Wollan in [4] gave an example of a countable family of *FK* spaces contained in m whose union is dense in m. They can be realized as summability domains in the following manner. Let $\{r_i\}$ be a denumeration of the nonzero rationals. Define $A_i = (a_{nk}^{(i)})$ by

(i) $a_{n1}^{(i)} = r_i$, $a_{n2}^{(i)} = -1$, $n = 1, 3, 5, \cdots$

(ii) $a_{n1}^{(i)} = -1$, $a_{n2}^{(i)} = r_i^{-1}$, $n = 2, 4, 6, \cdots$

$a_{nk} = 0$, $k \geqq 3$, $n = 1, 2, 3, \cdots$.

Then $\mathscr{A}_i = \{(x_n)_{n=1}^{\infty} \colon x_1 = x$, $x_2 = r_i x$, x_k arbitrary for $k \geqq 3$ and x complex$\} \cap m$. Each $\mathscr{A}_i$ is nowhere dense in m, but $\bigcup_{i=1}^{\infty} \mathscr{A}_i$ is dense. Note, however, that $\mathscr{A}_i \not\supset E^\infty$. Hence the hypothesis that each

$\mathscr{A}_i \supsetneqq E^\infty$ cannot be removed and our result is in some sense best possible.

Although we have proved our result only for $\mathscr{A}_i$, we conjecture that the following more general result holds:

Conjecture. If $\{F_i\}$ is a countable collection of FK-spaces each containing E^∞ but not m, then $\bigcup_{i=1}^{\infty} F_i$ is not dense in m. (See [8] for definitions and basic results.)

References

1. G. Bennett and N. J. Kalton, *FK-spaces containing* c_0, Duke Math. J., **39** (1972), 561–582.
2. S. R. Caradus, W. E. Pfaffenberger and Bertram Yood, *Calkin Algebras and Algebras of Operators on Banach Spaces*, Marcel Dekker, New York, 1974.
3. R. DeVos, *Category of sequences of* 0's *and* 1's *in some FK spaces*, to appear in the Glasgow Math. J.
4. C. Goffman and G. N. Wollan, *Sequences of regular summability matrices*, Monats. für Math., **76** (1972), 118–120.
5. I. J. Maddox, *Elements of Functional Analysis*, Cambridge Univ. Press, 1970.
6. G. M. Petersen, *Summability fields which span the bounded sequences densely*, Bull. London Math. Soc., **5** (1973), 187–191.
7. ———, Addendum: *Summability fields which span the bounded sequences densely*, Bull. London Math. Soc., **7** (1975), 105.
8. A. Wilansky, *Functional Analysis*, Blaisdell, New York, 1964.

Received April 4, 1977 and in revised form August 8, 1977.

Villanova University
Villanova, PA 19085